



A Survey of Probability Concepts

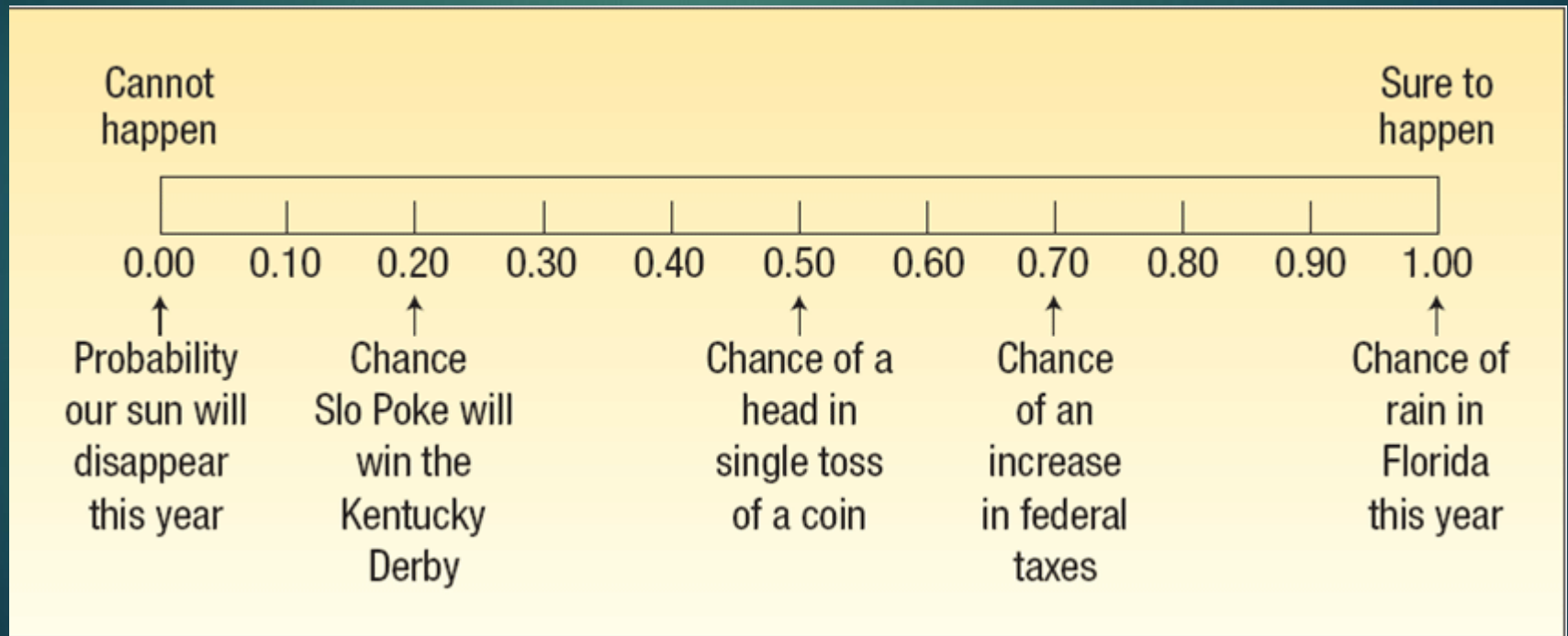
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GOALS

1. Define *probability*.
2. Describe the *classical, empirical, and subjective* approaches to probability.
3. Explain the terms *experiment, event, outcome, permutations, and combinations*.
4. Define the terms *conditional probability and joint probability*.
5. Calculate probabilities using the *rules of addition and rules of multiplication*.
6. Apply a *tree diagram* to organize and compute probabilities.
7. Calculate a probability using *Bayes' theorem*.



Probability

PROBABILITY A value between zero and one, inclusive, describing the relative possibility (chance or likelihood) an event will occur.



Experiment, Outcome and Event

- ▶ An **experiment** is a process that leads to the occurrence of one and only one of several possible observations.
- ▶ An **outcome** is the particular result of an experiment.
- ▶ An **event** is the collection of one or more outcomes of an experiment.

		
Experiment	Roll a die	Count the number of members of the board of directors for Fortune 500 companies who are over 60 years of age
All possible outcomes	Observe a 1 Observe a 2 Observe a 3 Observe a 4 Observe a 5 Observe a 6	None are over 60 One is over 60 Two are over 60 ... 29 are over 60 48 are over 60 ...
Some possible events	Observe an even number Observe a number greater than 4 Observe a number 3 or less	More than 13 are over 60 Fewer than 20 are over 60

Mutually Exclusive Events and Collectively Exhaustive Events

- ▶ Events are **mutually exclusive** if the occurrence of any one event means that none of the others can occur at the same time.
- ▶ Events are **independent** if the occurrence of one event does not affect the occurrence of another. Events are **collectively exhaustive** if at least one of the events must occur when an experiment is conducted.
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Ways of Assigning Probability

There are three ways of assigning probability:

1. CLASSICAL PROBABILITY

Based on the assumption that the outcomes of an experiment are *equally likely*.

2. EMPIRICAL PROBABILITY

The probability of an event happening is the fraction of the time similar events happened in the past.

3. SUBJECTIVE CONCEPT OF PROBABILITY

The likelihood (probability) of a particular event happening that is assigned by an individual based on whatever information is available.

Classical Probability

CLASSICAL PROBABILITY

$$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}} \quad [5-1]$$

Consider an experiment of rolling a six-sided die. What is the probability of the event “an **even number** of spots appear face up”?

The possible outcomes are:



There are three “favorable” outcomes (a two, a four, and a six) in the collection of six equally likely possible outcomes.

Empirical Probability

EMPIRICAL PROBABILITY The probability of an event happening is the fraction of the time similar events happened in the past.

The empirical approach to probability is based on what is called the law of large numbers. The key to establishing probabilities empirically is that more observations will provide a more accurate estimate of the probability.

LAW OF LARGE NUMBERS Over a large number of trials the empirical probability of an event will approach its true probability.

Empirical Probability - Example

On February 1, 2003, the Space Shuttle Columbia exploded. This was the second disaster in 113 space missions for NASA. On the basis of this information, what is the probability that a future mission is successfully completed?

$$\begin{aligned}\text{Probability of a successful flight} &= \frac{\text{Number of successful flights}}{\text{Total number of flights}} \\ &= \frac{111}{113} = 0.98\end{aligned}$$

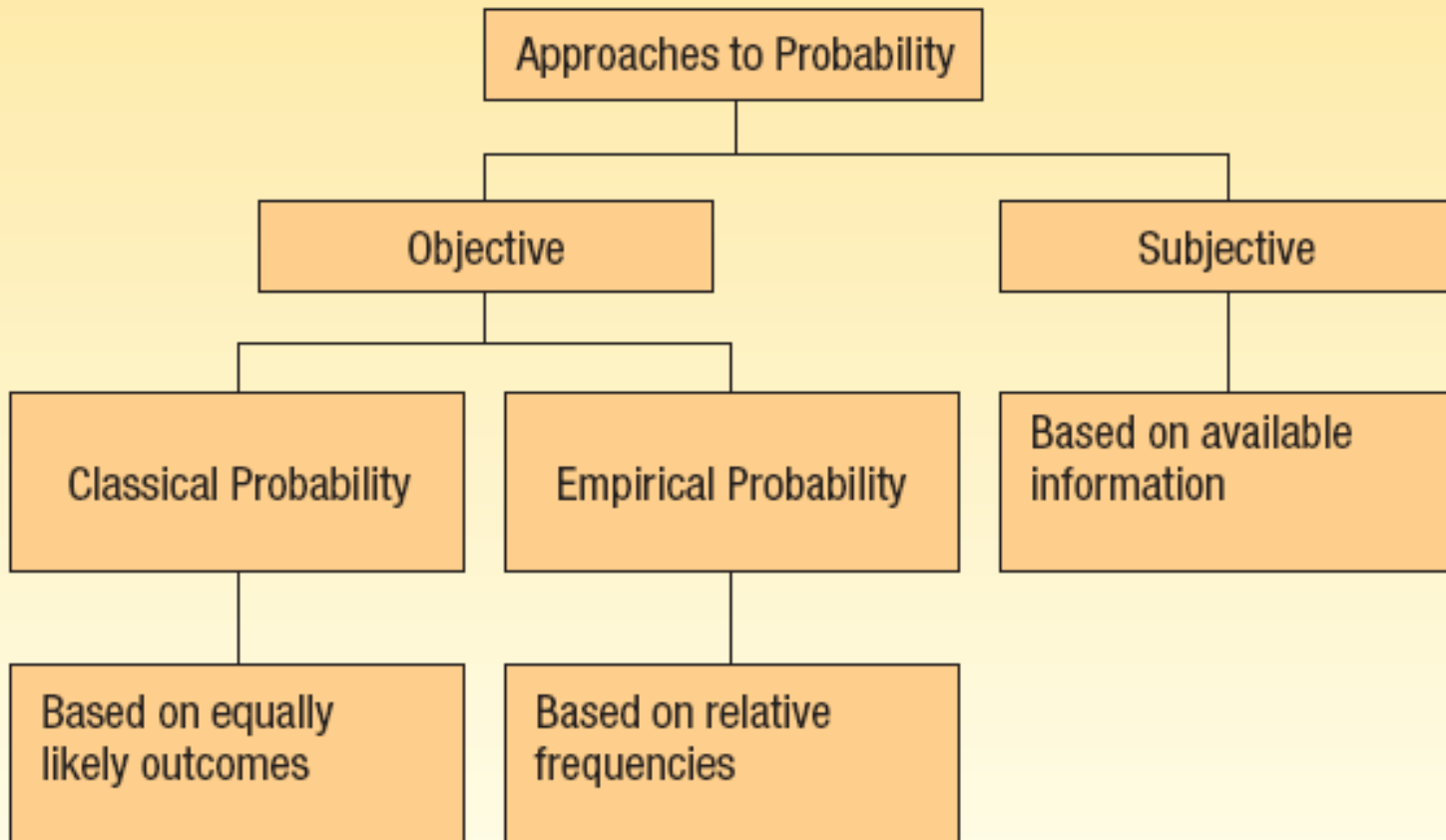
Subjective Probability - Example

SUBJECTIVE CONCEPT OF PROBABILITY The likelihood (probability) of a particular event happening that is assigned by an individual based on whatever information is available.

- ▶ If there is little or no past experience or information on which to base a probability, it may be arrived at subjectively.

- ▶ Illustrations of subjective probability are:
 1. Estimating the likelihood the New England Patriots will play in the Super Bowl next year.
 2. Estimating the likelihood you will be married before the age of 30.
 3. Estimating the likelihood the U.S. budget deficit will be reduced by half in the next 10 years.

Summary of Types of Probability



Rules for Computing Probabilities

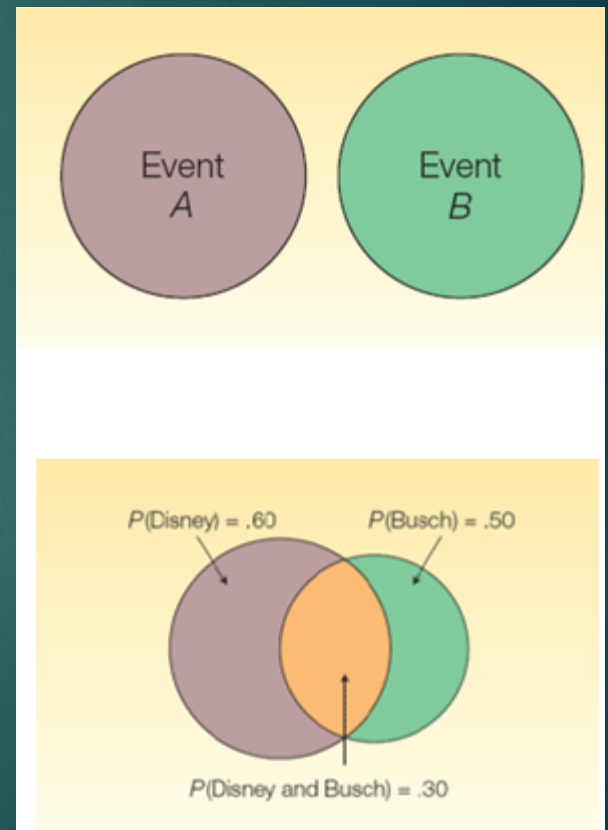
Rules of Addition

- ▶ Special Rule of Addition - If two events A and B are mutually exclusive, the probability of one or the other event's occurring equals the sum of their probabilities.

$$P(A \text{ or } B) = P(A) + P(B)$$

- ▶ The General Rule of Addition - If A and B are two events that are not mutually exclusive, then $P(A \text{ or } B)$ is given by the following formula:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$



Addition Rule – Mutually Exclusive Events Example



An automatic Shaw machine fills plastic bags with a mixture of beans, broccoli, and other vegetables. Most of the bags contain the correct weight, but because of the variation in the size of the beans and other vegetables, a package might be underweight or overweight. A check of 4,000 packages filled in the past month revealed:

What is the probability that a particular package will be either underweight or overweight?

$$P(A \text{ or } C) = P(A) + P(C) = .025 + .075 = .10$$

Weight	Event	Number of Packages	Probability of Occurrence	
Underweight	<i>A</i>	100	.025	← $\frac{100}{4,000}$
Satisfactory	<i>B</i>	3,600	.900	
Overweight	<i>C</i>	300	.075	
		<u>4,000</u>	<u>1.000</u>	

Addition Rule – Not Mutually Exclusive Events Example

What is the probability that a card chosen at random from a standard deck of cards will be either a king or a heart?

Card	Probability	Explanation
King	$P(A) = 4/52$	4 kings in a deck of 52 cards
Heart	$P(B) = 13/52$	13 hearts in a deck of 52 cards
King of Hearts	$P(A \text{ and } B) = 1/52$	1 king of hearts in a deck of 52 cards

$$= 4/52 + 13/52 - 1/52$$

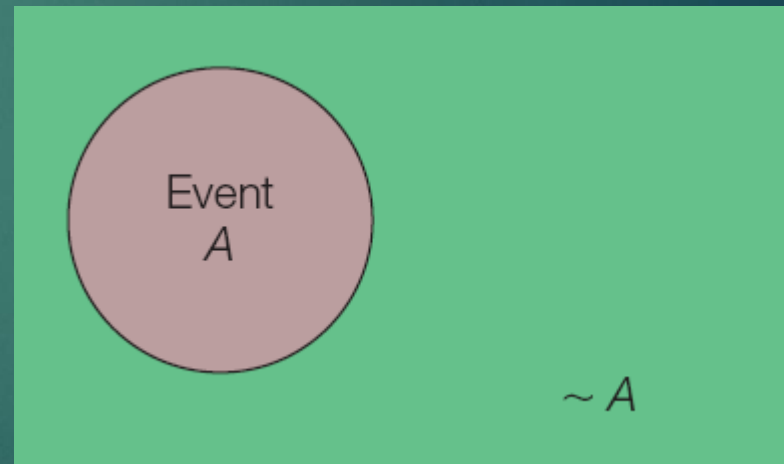
$$= 16/52, \text{ or } .3077$$

The Complement Rule

The **complement rule** is used to determine the probability of an event occurring by subtracting the probability of the event *not* occurring from 1.

$$P(A) + P(\sim A) = 1$$

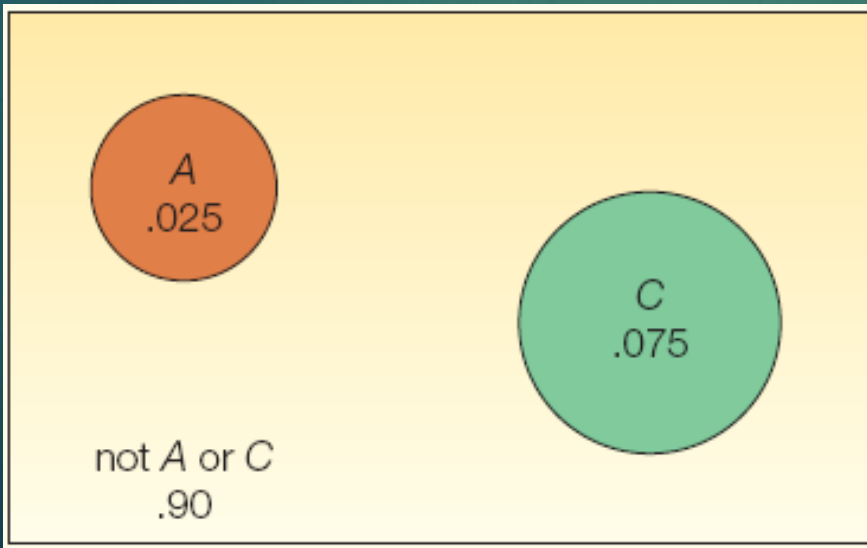
or $P(A) = 1 - P(\sim A)$.



The Complement Rule - Example

An automatic Shaw machine fills plastic bags with a mixture of beans, broccoli, and other vegetables. Most of the bags contain the correct weight, but because of the variation in the size of the beans and other vegetables, a package might be underweight or overweight. Use the complement rule to show the probability of a satisfactory bag is .900

Weight	Event	Number of Packages	Probability of Occurrence
Underweight	A	100	.025
Satisfactory	B	3,600	.900
Overweight	C	300	.075
		<u>4,000</u>	<u>1.000</u>



$$\begin{aligned} P(B) &= 1 - P(\sim B) \\ &= 1 - P(A \text{ or } C) \\ &= 1 - [P(A) + P(C)] \\ &= 1 - [.025 + .075] \\ &= 1 - .10 \\ &= .90 \end{aligned}$$

The General Rule of Addition

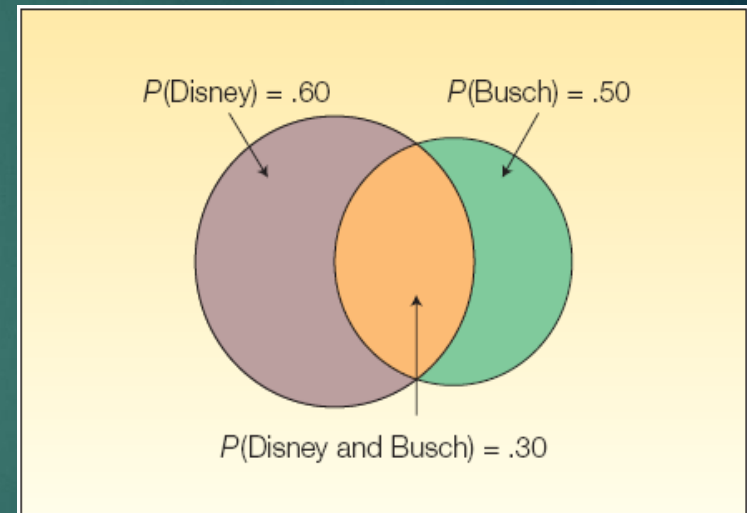
GENERAL RULE OF ADDITION

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

[5-4]

The Venn Diagram shows the result of a survey of 200 tourists who visited Florida during the year. The survey revealed that 120 went to Disney World, 100 went to Busch Gardens and 60 visited both.

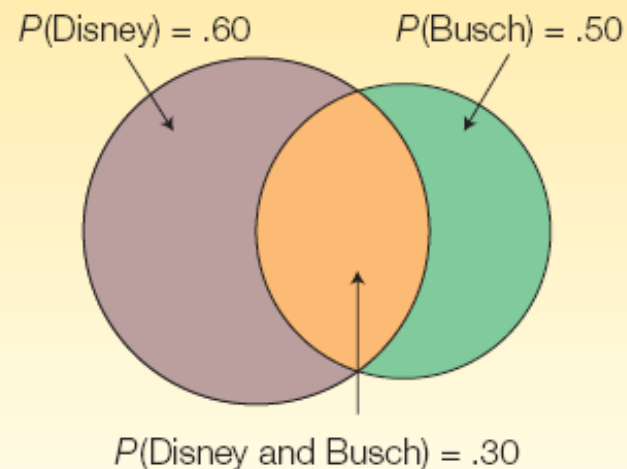
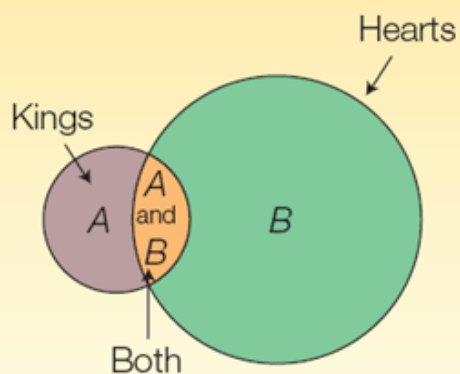
What is the probability a selected person visited either Disney World or Busch Gardens?



$$\begin{aligned} P(\text{Disney or Busch}) &= P(\text{Disney}) + P(\text{Busch}) - P(\text{both Disney and Busch}) \\ &= 120/200 + 100/200 - 60/200 \\ &= .60 + .50 - .30 \end{aligned}$$

Joint Probability – Venn Diagram

JOINT PROBABILITY A probability that measures the likelihood two or more events will happen concurrently.



Special Rule of Multiplication

- ▶ The **special rule of multiplication** requires that two events A and B are *independent*.
- ▶ Two events A and B are independent if the occurrence of one has no effect on the probability of the occurrence of the other.
- ▶ This rule is written: $P(A \text{ and } B) = P(A)P(B)$

Multiplication Rule-Example

A survey by the American Automobile association (AAA) revealed 60 percent of its members made airline reservations last year. Two members are selected at random. Since the number of AAA members is very large, we can assume that R_1 and R_2 are independent. What is the probability both made airline reservations last year?

Solution:

The probability the first member made an airline reservation last year is .60, written as $P(R_1) = .60$

The probability that the second member selected made a reservation is also .60, so $P(R_2) = .60$.

Since the number of AAA members is very large, you may assume that R_1 and R_2 are independent.

$$P(R_1 \text{ and } R_2) = P(R_1)P(R_2) = (.60)(.60) = .36$$

Conditional Probability

- ▶ A **conditional probability** is the probability of a particular event occurring, given that another event has occurred.
- ▶ The probability of the event A given that the event B has occurred is written $P(A | B)$.

General Multiplication Rule

The **general rule of multiplication** is used to find the joint probability that two *independent* events will occur.

It states that for two independent events, A and B , the joint probability that both events will happen is found by multiplying the probability that event A will happen by the conditional probability of event B occurring, given that A has occurred.

GENERAL RULE OF MULTIPLICATION

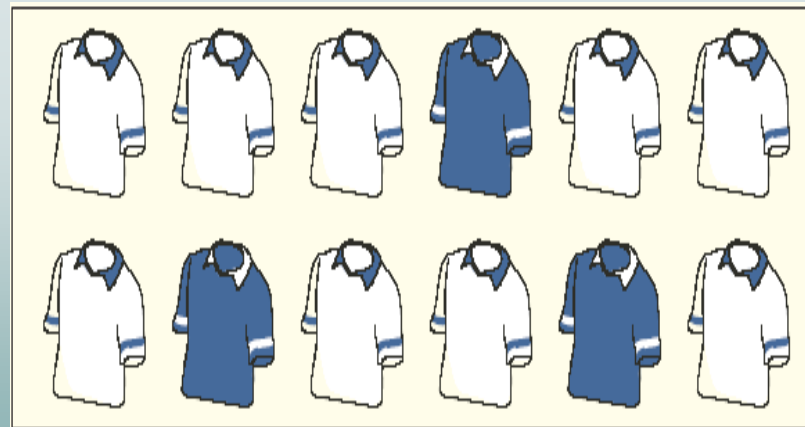
$$P(A \text{ and } B) = P(A)P(B|A)$$

[5-6]

General Multiplication Rule - Example

A golfer has 12 golf shirts in his closet. Suppose 9 of these shirts are white and the others blue. He gets dressed in the dark, so he just grabs a shirt and puts it on. He plays golf two days in a row and does not do laundry.

What is the likelihood both shirts selected are white?



- ▶ The event that the first shirt selected is white is W_1 . The probability is $P(W_1) = 9/12$
- ▶ The event that the second shirt (W_2) selected is also white. The conditional probability that the second shirt selected is white, given that the first shirt selected is also white, is $P(W_2 | W_1) = 8/11$.
- ▶ To determine the probability of 2 white shirts being selected we use formula: $P(AB) = P(A) P(B | A)$
- ▶ $P(W_1 \text{ and } W_2) = P(W_1)P(W_2 | W_1) = (9/12)(8/11) = 0.55$

Contingency Tables

A **CONTINGENCY TABLE** is a table used to classify sample observations according to two or more identifiable characteristics

E.g. A survey of 150 adults classified each as to gender and the number of movies attended last month. Each respondent is classified according to two criteria—the number of movies attended and gender.

Movies Attended	Gender		Total
	Men	Women	
0	20	40	60
1	40	30	70
2 or more	10	10	20
Total	70	80	150

Contingency Tables - Example

A sample of executives were surveyed about their loyalty to their company. One of the questions was, "If you were given an offer by another company equal to or slightly better than your present position, would you remain with the company or take the other position?" The responses of the 200 executives in the survey were cross-classified with their length of service with the company.

Loyalty	Length of Service				Total
	Less than 1 Year, B_1	1-5 Years, B_2	6-10 Years, B_3	More than 10 Years, B_4	
Would remain, A_1	10	30	5	75	120
Would not remain, A_2	25	15	10	30	80
	35	45	15	105	200

What is the probability of randomly selecting an executive who is loyal to the company (would remain) and who has more than 10 years of service?

Contingency Tables - Example

Event A_1 happens if a randomly selected executive will remain with the company despite an equal or slightly better offer from another company. Since there are 120 executives out of the 200 in the survey who would remain with the company

$$P(A_1) = 120/200, \text{ or } .60.$$

Event B_4 happens if a randomly selected executive has more than 10 years of service with the company. Thus, $P(B_4 | A_1)$ is the conditional probability that an executive with more than 10 years of service would remain with the company. Of the 120 executives who would remain 75 have more than 10 years of service, so $P(B_4 | A_1) = 75/120$.

$$P(A_1 \text{ and } B_4) = P(A_1)P(B_4 | A_1) = \left(\frac{120}{200}\right)\left(\frac{75}{120}\right) = \frac{9,000}{24,000} = .375$$

Tree Diagrams



A **tree diagram** is useful for portraying conditional and joint probabilities. It is particularly useful for analyzing business decisions involving several stages.

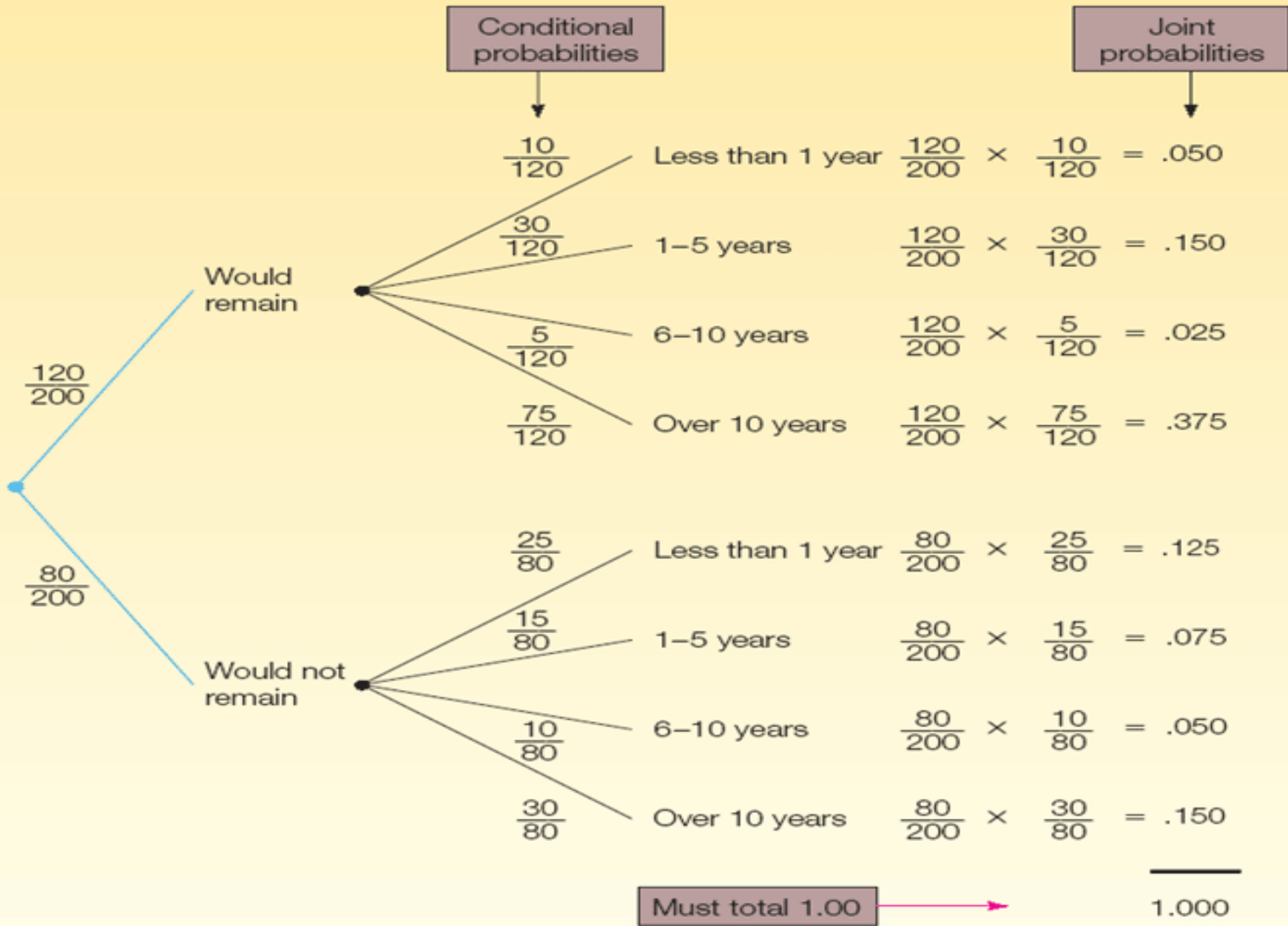
A **tree diagram** is a graph that is helpful in organizing calculations that involve several stages. Each segment in the tree is one stage of the problem. The branches of a tree diagram are weighted by probabilities.

Loyalty

Service

Conditional probabilities

Joint probabilities



Bayes' Theorem

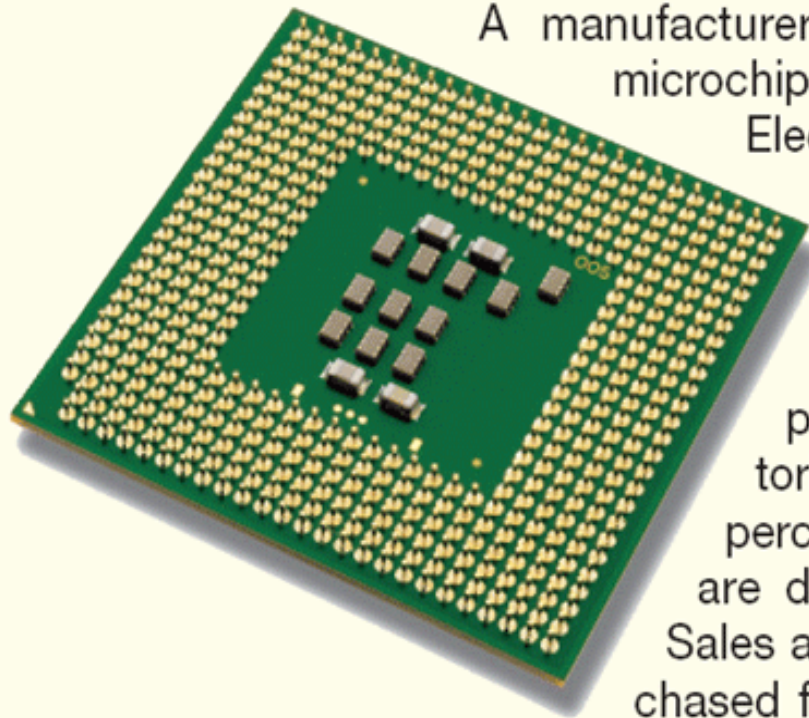
- ▶ Bayes' Theorem is a method for revising a probability given additional information.
- ▶ It is computed using the following formula:

BAYES' THEOREM

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

[5-7]

Bayes Theorem - Example



A manufacturer of DVD players purchases a particular microchip, called the LS-24, from three suppliers: Hall Electronics, Schuller Sales, and Crawford Components. Thirty percent of the LS-24 chips are purchased from Hall Electronics, 20 percent from Schuller Sales, and the remaining 50 percent from Crawford Components. The manufacturer has extensive histories on the three suppliers and knows that 3 percent of the LS-24 chips from Hall Electronics are defective, 5 percent of chips from Schuller Sales are defective, and 4 percent of the chips purchased from Crawford Components are defective.

When the LS-24 chips arrive at the manufacturer, they are placed directly in a bin and not inspected or otherwise identified by supplier. A worker selects a chip for installation in a DVD player and finds it defective. What is the probability that it was manufactured by Schuller Sales?

Bayes Theorem – Example (cont.)

- There are three mutually exclusive and collectively exhaustive events, that is, three suppliers.

A_1 The LS-24 was purchased from Hall Electronics.

A_2 The LS-24 was purchased from Schuller Sales.

A_3 The LS-24 was purchased from Crawford Components.

- The prior probabilities are:

$P(A_1) = .30$ The probability the LS-24 was manufactured by Hall Electronics.

$P(A_2) = .20$ The probability the LS-24 was manufactured by Schuller Sales.

$P(A_3) = .50$ The probability the LS-24 was manufactured by Crawford Components.

- The additional information can be either:

B_1 The LS-24 appears defective, or

B_2 The LS-24 appears not to be defective.

Bayes Theorem – Example (cont.)

- The following conditional probabilities are given.

$P(B_1|A_1) = .03$ The probability that an LS-24 chip produced by Hall Electronics is defective.

$P(B_1|A_2) = .05$ The probability that an LS-24 chip produced by Schuller Sales is defective.

$P(B_1|A_3) = .04$ The probability that an LS-24 chip produced by Crawford Components is defective.

- A chip is selected from the bin. Because the chips are not identified by supplier, we are not certain which supplier manufactured the chip. We want to determine the probability that the defective chip was purchased from Schuller Sales. The probability is written $P(A_2|B_1)$.

Bayes Theorem – Example (cont.)

Event, A_i	Prior Probability, $P(A_i)$	Conditional Probability, $P(B_1 A_i)$	Joint Probability, $P(A_i \text{ and } B_1)$	Posterior Probability, $P(A_i B_1)$
Hall	.30	.03	.009	$.009/.039 = .2308$
Schuller	.20	.05	.010	$.010/.039 = .2564$
Crawford	.50	.04	.020	$.020/.039 = .5128$
			$P(B_1) = .039$	1.0000

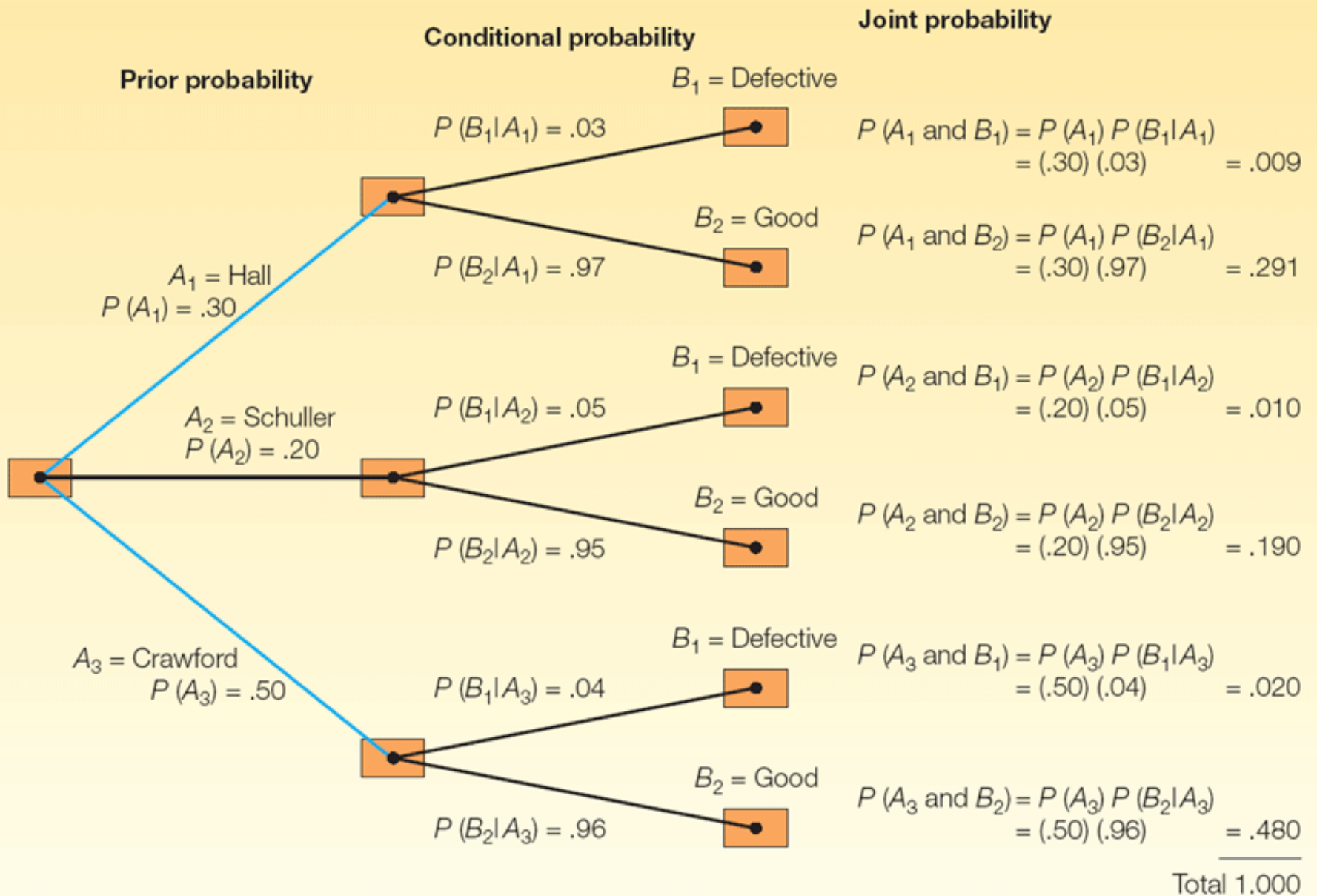


CHART 5-3 Tree Diagram of DVD Manufacturing Problem

Bayes Theorem – Example (cont.)

The probability the defective LS-24 chip came from Schuller Sales can be formally found by using Bayes' theorem. We compute $P(A_2|B_1)$, where A_2 refers to Schuller Sales and B_1 to the fact that the selected LS-24 chip was defective.

$$\begin{aligned} P(A_2|B_1) &= \frac{P(A_2)P(B_1|A_2)}{P(A_1)P(B_1|A_1) + P(A_2)P(B_1|A_2) + P(A_3)P(B_1|A_3)} \\ &= \frac{(.20)(.05)}{(.30)(.03) + (.20)(.05) + (.50)(.04)} = \frac{.010}{.039} = .2564 \end{aligned}$$

Counting Rules – Multiplication

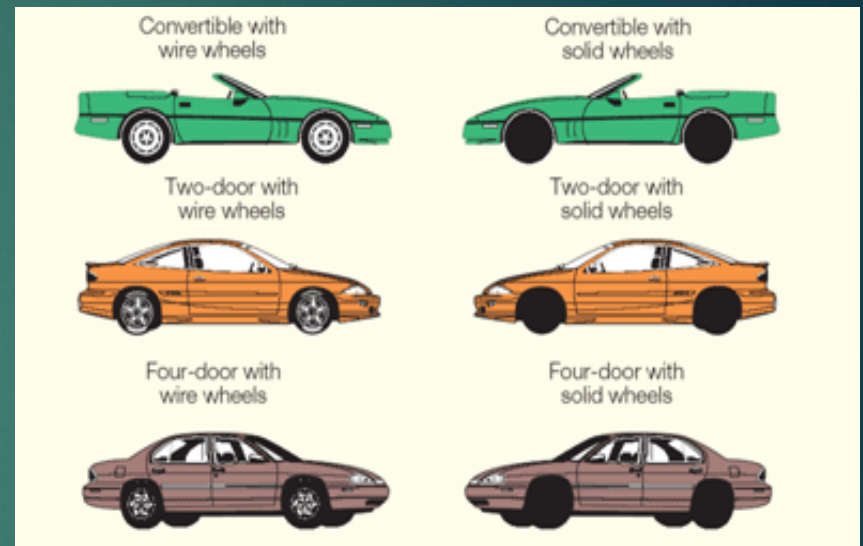
The **multiplication formula** indicates that if there are m ways of doing one thing and n ways of doing another thing, there are $m \times n$ ways of doing both.

Example: Dr. Delong has 10 shirts and 8 ties. How many shirt and tie outfits does he have?

$$(10)(8) = 80$$

Counting Rules – Multiplication: Example

An automobile dealer wants to advertise that for \$29,999 you can buy a convertible, a two-door sedan, or a four-door model with your choice of either wire wheel covers or solid wheel covers. How many different arrangements of models and wheel covers can the dealer offer?



MULTIPLICATION FORMULA

Total number of arrangements = $(m)(n)$

[5–8]

We can employ the multiplication formula as a check (where m is the number of models and n the wheel cover type). From formula (5–8):

$$\text{Total possible arrangements} = (m)(n) = (3)(2) = 6$$

Counting Rules - Permutation

A **permutation** is any arrangement of r objects selected from n possible objects. The order of arrangement is important in permutations.

PERMUTATION FORMULA

$${}_n P_r = \frac{n!}{(n-r)!}$$

[5-9]

where:

n is the total number of objects.

r is the number of objects selected.

Counting - Combination

A **combination** is the number of ways to choose r objects from a group of n objects without regard to order.

COMBINATION FORMULA

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

[5-10]

where:

n is the total number of objects.

r is the number of objects selected.

Combination and Permutation Examples

COMBINATION EXAMPLE

There are 12 players on the Carolina Forest High School basketball team. Coach Thompson must pick five players among the twelve on the team to comprise the starting lineup. How many different groups are possible?

$${}_{12}C_5 = \frac{12!}{5!(12-5)!} = 792$$

PERMUTATION EXAMPLE

Suppose that in addition to selecting the group, he must also rank each of the players in that starting lineup according to their ability.

$${}_{12}P_5 = \frac{12!}{(12-5)!} = 95,040$$